Lie group = group + (complex) manifold

$G \times G \to G$

tangent space @ identity

$\downarrow$

Lie algebra.

Eg. $G = GL_n(\mathbb{C}) \overset{\text{open}}{\subseteq} \mathbb{C}^{n^2}$ $G = SL_n$ other types affine

$\mathfrak{gl}_n(\mathbb{C}) = \underset{\text{matrices}}{\text{scalar}} \oplus \underset{\text{traceless matrices}}{\mathfrak{sl}_n}$

$G \curvearrowright G = GL_n$ or SL_n

conjugation

$= \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$

$G \curvearrowright T_e G = \mathfrak{g}$

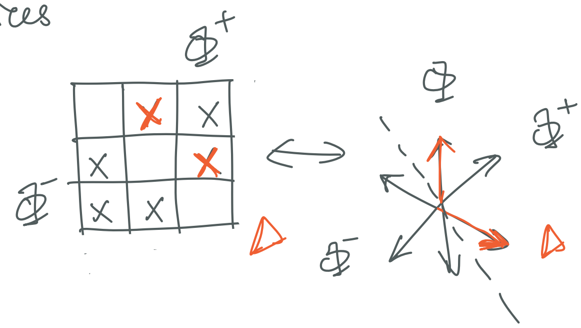
$\cup$

T.

diagonal matrices

Schur's Lemma

type A_{n-1}



Flag manifolds = compact manifolds with a transitive G-action

E.g. Riemann Sphere $\mathbb{CP}^1 = \{\dim 1 \text{ subspaces in } \mathbb{C}^2\}$

$= \{\text{span}(z,1) : z \in \mathbb{C}\} \cup \{\text{span}(1,0)\}$

∞

$\circlearrowright$

GL_2

$\mathbb{CP}^n = \{\dim 1 \text{ subspaces in } \mathbb{C}^{n+1}\} = Gr(1, n+1)$

$\circlearrowright$ GL_{n+1}

$Gr(m,n) = \{\dim m \text{ subspaces of } \mathbb{C}^n\}$ Ex. compact manifold

$\|$

$Fl(m,n)$

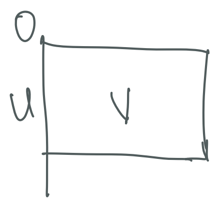
$\circlearrowright$ GL_n

ordered basis

$Fl(1,2,3) = \{\underset{\dim 1}{U} \subset \underset{\dim 2}{V} \subset \mathbb{C}^3\}$

$\circlearrowright$ GL_3 Complete flag manifold

partial



$\mathbb{CP}^2$.

$Fl(m_1, m_2, \ldots, m_\ell, n)$

$\circlearrowleft$

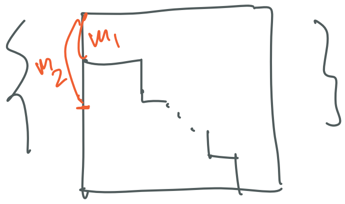
GL_n

SL_n/P

//

"orbit-stabilizer thm" $\Rightarrow X = GL_n/P$ ← parabolic subgroup

Ex. When $X = Fl(m_1, \ldots, m_\ell, n)$. P can be taken to be

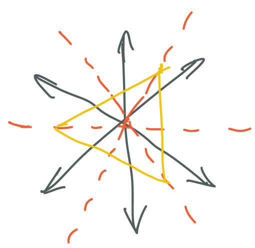


$\subseteq GL_n$.

E.g. $B := \left\{ \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix} \right\} \subseteq GL_n$, $GL_n/B = Fl(1, 2, \ldots, n-1, n)$.

$\Phi \rightsquigarrow$ Weyl group $W = \langle s_\alpha : \alpha \in \Phi \rangle = \langle s_\alpha : \alpha \in \Delta \rangle$

E.g. A_2 $W = S_3$



reduced word, length.

E.g. $u \in S_n$.

$$\ell(u) = \#\{(i,j) : i<j,\ u(i) > u(j)\}$$

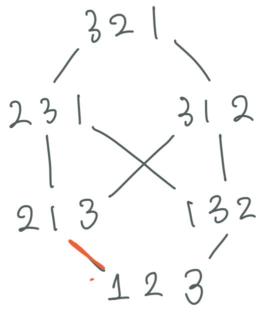
Bruhat order on W (a Coxeter group):

$u < v$ if $u = s_\alpha \cdot v$ for some $\alpha \in \Phi$ and $\ell(u) < \ell(v)$

+ transitive closure.

Bruhat graph. encodes geometry of G/B

E.g. S_3



vertices $\longleftrightarrow$ T-fixed points $\longleftrightarrow$ Schubert varieties := closures of B-orbits.

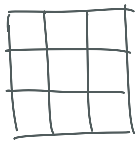
dim

edges $\longleftrightarrow$ T-stable curves

i.e. copies of $\mathbb{CP}^1 \subseteq G/P$ that are invariant under T-action

Pick a basis $\{e_1, e_2, e_3\}$ for $\mathbb{C}^3$

$X = Fl(1,2,3)$

 $\longrightarrow$ point in X.

T-action : left multiplication

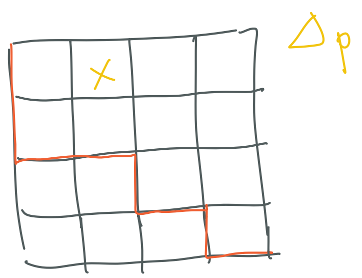
$\underbrace{u}_{V}$

What about G/P ?

Note :

$P \longleftrightarrow$ subset Δ_P of Δ

$= \{\alpha \in \Delta : \mathfrak{g}_{-\alpha} \subseteq T_eP\}$



$\Delta_B = \phi$.

$W_P = \langle s_\alpha : \alpha \in \Delta_P \rangle \subseteq W$.

$W_B = \{id\}$

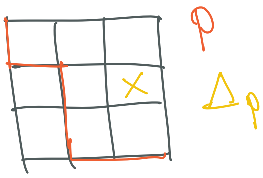
$W^P := \{$minimal representatives of $W/W_P\} \subseteq W$.

Bruhat order restricts

E.g.

$X = \mathbb{CP}^2 = Fl(1,3)$

T-fixed pts: $\left\{ \mathrm{span}\begin{pmatrix}1\\0\\0\end{pmatrix}, \mathrm{span}\begin{pmatrix}0\\1\\0\end{pmatrix}, \mathrm{Span}\begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$



$W/W_p = S_3/S_2$

W^p		Schubert varieties
312	$\longleftrightarrow$	$\mathbb{CP}^2$
$\vert$		$\cup$
213	$\longleftrightarrow$	$\mathbb{CP}^1$
$\vert$		$\cup$
123	$\longleftrightarrow$	pt.

Borel-Weil Theorem.

~~Recall that a representation of G is a finite dimensional $\mathbb{C}$-vector space V together with $\rho: G \to GL(V)$~~

~~morphism of Lie groups, i.e., group homomorphism & holomorphic~~

~~A 1-dimensional representation $G \to \mathbb{C}^*$ is called a character of G~~

Def. $\varepsilon_i : \mathrm{diag}(t_1, \dots, t_n) \mapsto t_i$

Ex. Characters of $T \leq Gl_n = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n$

Characters of $T \leq Sl_n = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n / \mathbb{Z}(\varepsilon_1 + \cdots + \varepsilon_n)$

~~A representation V is irreducible if there is no proper non-trivial G-invariant subspace.~~ semisimple.

Irreducible representations of $G \longleftrightarrow$ dominant weights of T.

Sl_n $\quad \{\text{character } \lambda : T \to \mathbb{C}^* \mid \lambda_1 \geq \cdots \geq \lambda_n\}$

λ extends to a character $\lambda : \underset{TU.}{\overset{}{B}} \to \mathbb{C}^*$ ($B = TU.$)

V is an irreducible representation of G iff there is a unique (up to scalar) $v \in V$ such that $\underbrace{b.v}_{\rho(b)(v)} = \underbrace{\lambda(b)}_{\text{heighest weight}} v \quad \forall b \in B$

OTOH, λ defines a line bundle on G/B:

$$\begin{array}{ccc} \text{total space } G \curvearrowright L_\lambda = G \times \mathbb{C}/\sim & & [g, z] \sim [gb, \lambda(b) z] \quad \forall b \in B. \\ \pi \downarrow & & \downarrow \\ G/B. & & gB \end{array}$$

$(G \times^B \mathbb{C}_{-\lambda})$

preimage .. fiber $\mathbb{C}$. Transition functions are linear isomorphisms
locally trivial.

E.g. $G = GL_3$ or SL_3

$$\begin{array}{c} L_\lambda \\ \pi \downarrow \end{array}$$

$G/B \supseteq \Omega := \{ U \subset V \subset \mathbb{C}^3 : U \not\subseteq \operatorname{span}\{e_1, e_2\},\ e_1 \notin V \} \simeq \mathbb{C}^3$

$$\updownarrow \quad \left\{ \begin{bmatrix} * & * & 1 \\ * & 1 & \\ 1 & & \end{bmatrix} \right\} \qquad \text{Ex. } \pi^{-1}(\Omega) \xrightarrow[\simeq]{\varphi} \Omega \times \mathbb{C}$$

equivariant ...

$\Omega' = \{ U \subset V \subset \mathbb{C}^3 : U \not\subseteq \operatorname{span}\{e_2, e_3\},\ e_3 \notin V \}$

$$\updownarrow \quad \left\{ \begin{bmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{bmatrix} \right\} \qquad \pi^{-1}(\Omega') \xrightarrow[\simeq]{\psi} \Omega' \times \mathbb{C}$$

Ex. $(\Omega \cap \Omega') \times \mathbb{C} \xrightarrow[\simeq]{\psi \circ \varphi^{-1}} (\Omega \cap \Omega') \times \mathbb{C}$ has the form

$$(y, z) \longmapsto (y, f(y) z)$$

$$f : \Omega \cap \Omega' \to \mathbb{C}^*$$

A section of L_λ is $\begin{array}{c} L_\lambda \\ s \uparrow \\ G/B \end{array}$ such that $\pi \circ s = id$.

Space of sections

$$H^0(G/B, L_\lambda) = \{\tilde{s} : G \to \mathbb{C} \text{ holomorphic} \mid \tilde{s}(gb) = \lambda(b)\,\tilde{s}(g),\ \forall g \in G, b \in B\}$$

$$\circlearrowleft G \qquad \updownarrow$$

$$s : gB \mapsto [g, \tilde{s}(g)]$$

$$g.\tilde{s} = \tilde{s} \circ g^{-1}$$

Borel–Weil Theorem:

$H^0(G/B, L_\lambda)$ is the irreducible G representation with highest weight $-w_0(\lambda)$.

E.g. $G = SL_3$, $\lambda = \varepsilon_1$

$$\tilde{s} : g \mapsto g_{11}$$

$$\tilde{s}(gb) = b_{11} g_{11} = \varepsilon_1(b)\,\tilde{s}(g) \checkmark$$

$$\forall\, b^- \in B^-, \quad b^-.\tilde{s}(g) = \tilde{s}(b^{-1}g) = (b^-_{11})^{-1} g_{11} = -\varepsilon_1(b^-)\,\tilde{s}(g)$$

$$\begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$$

$$B = w_0 B^- w_0^{-1}$$

$$\begin{aligned} b.w_0.\tilde{s} &= w_0 b^- w_0^{-1}.w_0\tilde{s} \\ &= w_0 b^-.\tilde{s} \\ &= w_0(-\varepsilon_1(b^-)\tilde{s}) \\ &= -\varepsilon_1(b^-)\, w_0.\tilde{s} \\ &= -\varepsilon_1(w_0^{-1} b w_0)\, w_0\tilde{s} \\ &= -w_0.\varepsilon_1(b)\, w_0.\tilde{s} \end{aligned}$$

$$\tilde{f} := w_0.\tilde{s} : g \mapsto g_{31}$$

$$\begin{aligned} b.\tilde{f}(g) &= \tilde{f}(b^{-1}.g) \\ &= b_{33}^{-1} g_{31} \\ &= -\varepsilon_3(b)\,\tilde{f}(g) \end{aligned}$$

Borel-Weil-Bott. Weyl character formula.